S-Parameter Matrices

The objective of microwave circuit analysis is to move from the requirement to solve for all the fields and waves of a structure to an equivalent circuit that is amenable to all the tools of the circuit analysis toolbox. However, the tools that are appropriate for lumped circuits must be extended to apply to distributed networks.

A matrix that is of great use in microwave network problems is the "scattering" matrix, so-called by analogy to the scattering or reflection of waves by a free-space reflector. As introduced in the prior notes, S-parameters have become the preferred description of microwave n-ports for the following reasons:

Voltage and current are difficult to define and measure in distributed circuits

The measurement of power in incident and reflected waves is a natural technique for microwave transmission lines. Voltage and current may not be well defined, or even defined at all, in some structures. The specification of voltage and current in a distributed circuit requires a specification of the exact location, and these parameters vary with location in the circuit. The determination of the individual parameters of voltage and current equation sets requires short or open circuit loads, which are sensitive to the precise location; in particular, it is not practical to mount a connector close enough to a microwave lumped device to be measuring its actual port voltages and currents. Also, many active devices cannot be operated with fully reflective terminations (short or open) of arbitrary phase, as they will oscillate, which is a large signal nonlinear condition and may even result in device failure.

Incident and reflected waves are the natural description for microwave structures

The matched condition ($\Gamma = 0$) is a unique, repeatable termination. It is insensitive to the length of transmission line to the matched load, so that measurements can be made without requiring the reference planes (the port connectors) to be located directly at the device under measurement (or being described). A matched load is a natural structure that can maintain its character over a very broad frequency range.

Conversion from S-parameters to other parameter sets is a matter of routine algebra

Each of the many equivalent parameter sets is uniquely useful for a given circuit topology. For example, the ABCD and T matrices are adept at cascaded networks, while the Z and Y parameter sets can be directly evaluated for tee and pi networks, respectively. However, the conversion process, while complex appearing, lends itself to repeated routines for hand or computer calculation.

S-parameters (in fact, all the parameter sets) benefit from the matrix toolbox.

The toolbox of established matrix mathematics is directly applicable to the matrices that are the equivalent of the port equations of the parameter sets. For example, the S matrix can be inspected for lossless, reciprocal or unilateral character. If either or both of these conditions is present, many of the individual matrix elements can be determined by inspection.

Equivalence of Matrix and Equation Form

For a single port network, we have the following simple relationships from our study of Γ and Smith chart.

$$b_1 = \Gamma a_1 = S_{11}a_1 \qquad\qquad\qquad b_1 = \Gamma a_1 = S_{11}a_1$$

For a multiport network the reflection coefficient is Γ defined as

$b_n = \Gamma_n a_n$, so $\Gamma_n = \dfrac{b_n}{a_n}$ where n is the port number.

Note that $\Gamma_n = S_{nn}$ <u>only</u> if all other ports are terminated, that is, only if all $a_m = 0$ for m?n. Otherwise it must be algebraically calculated from all the parameters.

The example of 2-port equations and their equivalent matrix is shown here to emphasize that both forms contain the same information, but the matrix form suggests the use of formal matrix algebra tools as an aid to analysis:

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \qquad\qquad \begin{aligned} b_1 &= S_{11}a_1 + S_{12}a_2 \\ b_2 &= S_{21}a_1 + S_{22}a_2 \end{aligned}$$

It is not uncommon to consider microwave networks of three and four ports, as in power dividers and directional couplers. The extension to the example of 3-port equations and equivalent matrix should reinforce the concept of equivalence:

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad\qquad \begin{aligned} b_1 &= S_{11}a_1 + S_{12}a_2 + S_{13}a_3 \\ b_2 &= S_{21}a_1 + S_{22}a_2 + S_{23}a_3 \\ b_3 &= S_{31}a_1 + S_{32}a_2 + S_{33}a_3 \end{aligned}$$

Using matrix rules we can inspect an S-parameter matrix and see whether the network is reciprocal and whether it is lossless. We can then use these facts to reduce the number of independent variables in the matrix, so that we can more easily evaluate the matrix elements (the parameters of the equations).

Which Matrices Are Used When?

We've seen that the equations and related matrix expressions all involve various combinations of a_n, b_n, V_n, and I_n. Certain of the forms are more easily evaluated in specific network topologies.

For example, suppose the network is composed of impedances as here
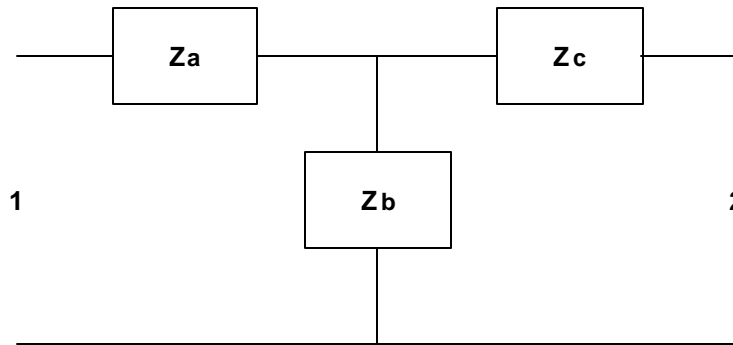
If we considering the impedance at port one for an open-circuited port 2, we see that

$Z_{11} = Z_a + Z_b$ and similarly $Z_{22} = Z_c + Z_b$

.
If we apply input current I_1 the open circuit output voltage is $I_1 Z_b$, so $Z_{12} = Z_b$,.

Solving for Z_{11}, Z_{12} and Z_{22} we find

$Z_{11} - Z_{12} = Z_a$

$Z_{22} - Z_{12} = Z_c$ and

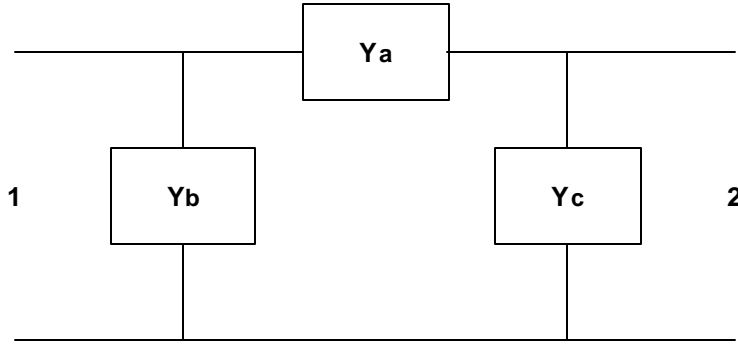$Z_{12} = Z_{21} = Z_b$

This defines the parameters of the Z matrix description

$$[V] = [Z][I]$$
$$V_1 = Z_{11} I_1 + Z_{12} I_2$$
$$V_2 = Z_{21} I_1 + Z_{22} I_2$$

The T network lends itself to the Z matrix description, which can then be converted to the equivalent S matrix for S-parameter measurements and operations.

Now consider the p form of a network, shown here:

If we consider the admittance at port 1 for shorted port 2, we see that for this network

$Y_{11} + Y_{12} = Y_a$; similarly

$Y_{22} + Y_{12} = Y_c$ and

$Y_{12} = -Y_b$

$$[I] = [Y][V]$$

$$I_1 = Y_{11} V_1 + Y_{12} V_2$$
$$I_2 = Y_{21} V_1 + Y_{22} V_2$$

Now consider the application of the ABCD matrix. The fact that the output voltage and current of the first of two cascaded networks are equal to the input voltage and the negative of the input current, respectively, of the second network makes the ABCD matrix a natural choice because is explicitly deals with the parameters V_n and I_n.
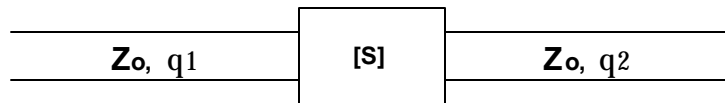
$$V_1 = A V_2 + B I_2$$
$$I_1 = C V_2 + D I_2$$
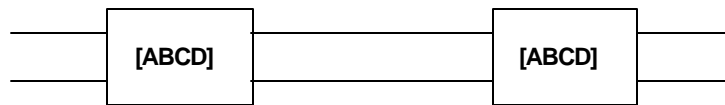
So by matrix multiplication, we can find the ABCD description of cascaded networks.

A last consideration is the question of a shift in reference planes, which is handled well by the S matrix. If we ask for the S matrix description of the following network

we find that it is simply $[S'] = [\theta_1][S][\theta_2]$, where $[\theta_n]$ is defined such that all terms are zero except the diagonal terms, which are $e^{-j2\theta_n}$.

Useful Matrix Operations

Certain simple matrix operations are useful in manipulating and evaluating S-parameter matrices. They include

- Multiplication (cascade ABCD or shift of reference plane
- Test for reciprocity
- Test for losslessness
- Test for unilateral transmission

Multiplication of matrices can be used to determine the ABCD or S parameters of cascaded networks of simpler forms.

The test for reciprocity requires that the matrix be symmetric, that is $S_{mn}=S_{nm}$. This can generally be determined by inspection.

The test for losslessness is that the sum of the $S_{mn}S_{mn}{}^*$ of any column must be unity. If the network is reciprocal, the matrix is symmetric and the same can be said of any row.

The test for unilateral transmission is that $S_{12} = 0$.

ABCD Example: Quarter- and Half-Wave Transmission Lines

The usefulness of ABCD parameters can be seen in an example that has been the subject of a homework problem. Consider a transmission line of length l.
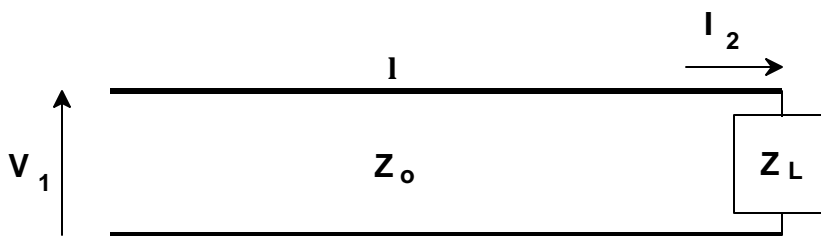
The ABCD parameters of this network relate V_i and I_i, such that

$V_1 = A\ V_2 + B\ I_2$, and

$I_1 = C\ V_2 + D\ I_2$.

The ABCD matrix of a length of transmission line l of Z_o and ß is

$A = \cos ßl$ $B = j Z_o \sin ßl$

$C = j Y_o \sin ßl$ $D = \cos ßl$

Note that, for $\cos ßl = 0$ (that is, for ßl=p/2, a quarter wavelength or odd multiple) the ABCD matrix becomes simply

$A = 0$ $B = j Z_o$

$C = j Y_o$ $D = 0$

which implies that $I_2 = V_1/j Z_o$ independent of V_2 or I_1. The fact that the output current depends only upon the input voltage and the characteristic impedance (regardless of the load impedance) is useful to construct feed structures for phased antennas, in which the element currents are the key parameters. If it desired to control the currents of several loads of varying impedance, each can be fed through identical quarter-wave lines that are paralleled at their inputs, thus insuring that, since they all have the same V_1 they all have the same I_2.

Similarly, if $\sin ßl = 0$ (that is, for ßl=p, a half wavelength or multiple) the ABCD matrix becomes simply

$A = -1$ $B = 0$

$C = 0$ $D = -1$

which implies that $V_2 = -V_1$ and $I_2 = -I_1$ independent of the terminating impedance at end 2.

Review of Transmission Lines

For the purpose of characterizing microwave multiport networks, key transmission line concepts are

      1)   Traveling waves in both directions, V^+ and V^-

      2)   Characteristic impedance Z_o and propagation constant jß

      3)   Reflection coefficient $\Gamma = \dfrac{Z_L - Z_o}{Z_L + Z_o}$ for complex load Z_L

4)   Standing waves resulting from $\Gamma ? 0$

5)   Transformation of Z_L through line of Z_o and length βl

6)   Description of Γ and Z on the Smith chart (polar graph of Γ)

Review of Scattering Matrix

1)   Normalization with respect to $\sqrt{Z_o}$ of wave amplitudes:

$$a = \frac{V^+}{\sqrt{Z_o}} \text{ and } b = \frac{V^-}{\sqrt{Z_o}} \text{, so power is aa* and bb*}$$

2)   Relationship of b_n and a_n: $b_n = \Gamma_n a_n$

3)   Expressions for b_1 and b_2 at reference planes:

$b_1 = S_{11}a_1 + S_{12}a_2$

$b_2 = S_{21}a_1 + S_{22}a_2$

4)   Definitions of S_{ii}:

$S_{11} = \dfrac{b_1}{a_1}$ for $a_2 = 0$, i.e., input Γ for output terminated in Z_o.

$S_{21} = \dfrac{b_2}{a_1}$ for $a_2 = 0$, i.e., forward transmission ratio with Z_o load.

$S_{22} = \dfrac{b_2}{a_2}$ for $a_1 = 0$, i.e., output Γ for input terminated in Z_o.

$S_{21} = \dfrac{b_1}{a_2}$ for $a_1 = 0$, i.e., reverse transmission ratio with Z_o source.

$|S_{21}|^2$ = Transducer power gain with Z_o source and load.

5)   Definitions of Γ_L, Γ_s, Γ_{in} and Γ_{out}:

$\Gamma_L = \dfrac{Z_L - Z_o}{Z_L + Z_o}$, the reflection coefficient of the load

$\Gamma_s = \dfrac{Z_s - Z_o}{Z_s + Z_o}$, the reflection coefficient of the source

$\Gamma_{in} = \dfrac{Z_{in} - Z_o}{Z_{in} + Z_o} = S_{11} + \dfrac{S_{12}S_{21}\Gamma_L}{1 - S_{22}\Gamma_L}$, the input reflection coefficient

$\Gamma_{out} = \dfrac{Z_{out} - Z_o}{Z_{out} + Z_o} = S_{22} + \dfrac{S_{12}S_{21}\Gamma_s}{1 - S_{11}\Gamma_s}$, the output reflection coefficient

6)   Power Gain G, Available Gain G_A, Transducer Gain G_T:

$$G = \frac{P_L}{P_{in}} = \frac{\text{power delivered to the load}}{\text{power input to the network}}$$

$$G_A = \frac{P_{avout}}{P_{avs}} = \frac{\text{power available from the network}}{\text{power available from the source}}$$

$$G_T = \frac{P_L}{P_{avs}} = \frac{\text{power delivered to the load}}{\text{power available from the source}}$$

Power Gain Equations

The equations for the various power gain definitions are

1) $\quad G = \dfrac{P_L}{P_{in}} = \dfrac{1}{1 - |\Gamma_{in}|^2} \ |S_{21}|^2 \dfrac{1 - |\Gamma_L|^2}{|1 - S_{22}\Gamma_L|^2}$

2) $\quad G_A = \dfrac{P_{avout}}{P_{avs}} = \dfrac{1 - |\Gamma_s|^2}{|1 - S_{11}\varpi_s|^2} \ |S_{21}|^2 \dfrac{1}{1 - |\Gamma_{out}|^2}$

3) $\quad G_T = \dfrac{P_L}{P_{avs}} = \dfrac{1 - |\Gamma_s|^2}{|1 - \Gamma_{in}\Gamma_s|^2} \ |S_{21}|^2 \dfrac{1 - |\Gamma_L|^2}{|1 - S_{22}\Gamma_L|^2}$

$$\qquad\qquad = \dfrac{1 - |\Gamma_s|^2}{|1 - S_{11}\Gamma_s|^2} \ |S_{21}|^2 \dfrac{1 - |\Gamma_L|^2}{|1 - \Gamma_{out}\Gamma_L|^2}$$

The expressions for Γ_{in} and Γ_{out} are

1) $\quad \Gamma_{in} = S_{11} + \dfrac{S_{12}S_{21}\Gamma_L}{1 - S_{22}\Gamma_L}$

2) $\quad \Gamma_{out} = S_{22} + \dfrac{S_{12}S_{21}\Gamma_s}{1 - S_{11}\Gamma_s}$

For a unilateral network, $S_{12}=0$ and

1) $\quad \Gamma_{in} = S_{11}$ if $S_{12}=0$ (unilateral network)

2) $\quad \Gamma_{out} = S_{22}$ if $S_{12}=0$ (unilateral network)

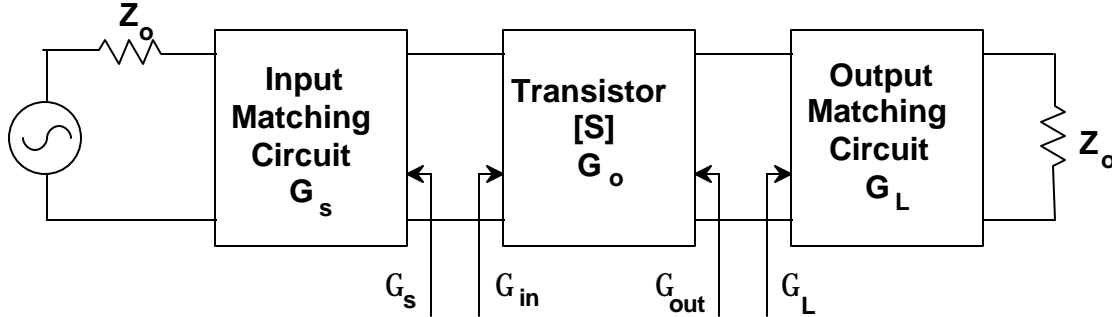The transducer gain G_T can be expressed as the product of three gain contributions

$G_T = G_s G_o G_L$, where

$G_o = |S_{21}|^2$

$G_s = \dfrac{1 - |\Gamma_s|^2}{|1 - \Gamma_{in}\Gamma_s|^2}$ and

$$G_L = \frac{1 - |\Gamma_L|^2}{|1 - S_{22}\Gamma_L|^2}$$

If the device is unilateral, or sufficiently so that S_{12} is small enough to be ignored, the unilateral transducer gain G_{TU} is simplified because

$$G_{sU} = \frac{1 - |\Gamma_s|^2}{|1 - S_{11}\Gamma_s|^2}$$, where the subscript U indicates unilateral gain.

In practice, the difference between G_T and G_{TU} is often quite small, as it is desirable for devices to be unilateral if possible.

The components of G_{TU} can also be expressed in decibel form, so that

$$G_{TU} \text{ (dB)} = G_s \text{ (dB)} + G_o \text{ (dB)} + G_L \text{ (dB)}.$$

We can maximize G_s and G_L by setting $\Gamma_s = S_{11}*$ and $\Gamma_L = S_{22}*$ so that

$$G_{s\,max} = \frac{1}{1 - |S_{11}|^2} \quad \text{and}$$

$$G_{L\,max} = \frac{1}{1 - |S_{22}|^2} \text{ , so that}$$

$$G_{TU\,max} = \frac{1}{1 - |S_{11}|^2} \, |S_{21}|^2 \, \frac{1}{1 - |S_{22}|^2}$$

Note that, if $|S_{11}|=1$ or $|S_{22}|=1$, $G_{TU\,max}$ is infinite. This raises the question of stability, which will be examined when we discuss amplifiers and oscillators.

Review of General Scheme of Solving Microwave Problems

n-port Equations	→	Equivalent Circuit S-parameters	← Matrix Tools
Losses, Fringing, Lumped Elements	→	Equivalent Transmission Line Z_o, α	← Smith Chart
Solve Equations for k_c	→	k_o, k_c, β	→ Free space Propagation, Antennas
Maxwell's Equations → $e^{j(\omega t - \beta z)}$ → Assume TEM, TE, TM Match Boundaries → Fields, Waves			

Coordinate System

Physical Structure